Fluctuations, strict convexity and scaling relation in first passage percolation

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More about total variation distance

Reminder: μ, ν prob. meas. on some set,

$$d_{TV}(\mu,\nu) = \sup_{A \text{ event}} |\mu(A) - \nu(A)|.$$

Claim: If μ and ν are abs. cont. wrt. a measure λ (e.g., Lebesgue or counting), and $\mu = f\,d\lambda$, $\nu = g\,d\lambda$, then

$$d_{TV}(\mu,\nu) = \frac{1}{2}\int |f-g|\,d\lambda = \int (f-g)1_{f>g}\,d\lambda = \int (g-f) 1_{g>f}\,d\lambda.$$

How to bound $d_{TV}(\mu,\nu)$ from above?

Instead of a direct estimate of the above integrals, can use two other distances.

Pinsker's inequality: $d_{TV}(\mu,\nu) \le \sqrt{\frac{1}{2} d_{KL}(\mu,\nu)}$

where d_{KL} is the Kullback-Leibler divergence, $d_{KL}(\mu,\nu) := \mathbb{E}_\mu\left(\log\left(\frac{d\mu}{d\nu}\right)\right) = \int \log\left(\frac{d\mu}{d\nu}\right) d\mu.$

assuming μ is abs. cont. wrt. ν

This is not quite a metric, since it is non-symmetric, $d_{KL}(\mu,\nu) \neq d_{KL}(\nu,\mu)$ in general. Has to do with entropy and information theory.

Proof is not long.

... inequality when ν is atomic,

Proof is not long.
(in other direction, when ν is atomic,

$$d_{TV}(\mu,\nu)^2 \geq \frac{1}{2} \min_{x:\, \nu(x)>0} \nu(x)\, d_{KL}(\mu,\nu)$$
)

Inequality with Hellinger distance:

Hellinger distance:

$$H^2(\mu,\nu) := \frac{1}{2}\int (\sqrt{f}-\sqrt{g})^2 d\lambda = 1 - \int \sqrt{f\cdot g}\, d\lambda$$

where, again, $\mu = f\,d\lambda$, $\nu = g\,d\lambda$.

$H(\mu,\nu)$ is a metric on prob. dist.
(it doesn't depend on the choice of λ)
satisfying $0 \leq H(\mu,\nu) \leq 1$.

Inequality: $H^2(\mu,\nu) \leq d_{TV}(\mu,\nu) \leq \sqrt{2}\, H(\mu,\nu)$.

Proof: $\leq$: $d_{TV}(\mu,\nu) = \frac{1}{2}\int |f-g|\, d\lambda =$

$$= \frac{1}{2}\int (\sqrt{f}-\sqrt{g})(\sqrt{f}+\sqrt{g})\, d\lambda \leq \text{Cauchy-Schwarz}$$

$$\leq \underbrace{\frac{1}{2}\sqrt{\int(\sqrt{f}-\sqrt{g})^2 d\lambda}}_{=\frac{1}{\sqrt{2}}H(\mu,\nu)} \cdot \sqrt{\underbrace{\int(\sqrt{f}+\sqrt{g})^2 d\lambda}_{=\int (f+g+2\sqrt{fg})d\lambda = 2+2\int\sqrt{fg}\,d\lambda}}$$

Cauchy-Schwarz $\to \leq 4$

$\geq$: $H^2(\mu,\nu) = \frac{1}{2}\int(\sqrt{f}-\sqrt{g})^2 d\lambda \leq$

$$\leq \frac{1}{2}\int |\sqrt{f}-\sqrt{g}|\cdot|\sqrt{f}+\sqrt{g}|\, d\lambda = \frac{1}{2}\int |f-g|\, d\lambda = d_{TV}(\mu,\nu).$$

Example: $X_1, \ldots, X_n$ indep. $\mathrm{Ber}(\frac{1}{2})$,
$(\mathbb{P}(X_i = 0) = \mathbb{P}(X_i = 1) = \frac{1}{2})$

$Y_1, \ldots, Y_n$ indep., with $Y_i \sim \mathrm{Ber}(\frac{1}{2}+\delta_i)$
$(\mathbb{P}(Y_i = 1) = \frac{1}{2}+\delta_i$, $\mathbb{P}(Y_i = 0) = \frac{1}{2}-\delta_i)$

$\ldots, n$

$(\mathbb{P}(Y_i = 1) = \frac{1}{2} + \delta_i, \ \mathbb{P}(Y_i = 0) = \frac{1}{2} - \delta_i)$

with all $\delta_i < c$ for some small c.

Then $d_{TV}(\mathcal{L}(X), \mathcal{L}(Y)) \leq C \|\delta\|_2$ for some $C > 0$.

using 1-$\sqrt{fg}\,d\lambda$ formula, with λ the counting measure on $\{0,1\}^n$.

Proof: $H^2(\mathcal{L}(X), \mathcal{L}(Y)) =$

$$1 - \sum_{x \in \{0,1\}^n} \sqrt{\frac{1}{2^n}} \cdot \sqrt{\prod_{i=1}^n \left(\frac{1}{2} + (-1)^{1-x_i}\delta_i\right)} =$$

$$= 1 - \frac{1}{2^n} \sum_{x \in \{0,1\}^n} \sqrt{\prod_{i=1}^n (1 + 2(-1)^{1-x_i}\delta_i)} \leq$$

$\sqrt{1-\varepsilon} = 1 - \frac{1}{2}\varepsilon - O(\varepsilon^2)$ for small ε

$$\leq 1 - \frac{1}{2^n} \sum_{x \in \{0,1\}^n} \prod_{i=1}^n (1 + (-1)^{1-x_i}\delta_i - C\delta_i^2) =$$

$$= 1 - \frac{1}{2^n} \sum_{x \in \{0,1\}^n} \prod_{i=1}^n (1 - C'\delta_i^2) = 1 - \prod_{i=1}^n (1 - C'\delta_i^2) \leq C'' \sum_{i=1}^n \delta_i^2$$

exercise, using that all δ_i are small.

This implies $d_{TV}(\mathcal{L}(X), \mathcal{L}(Y)) \leq C'' \|\delta\|_2$.

Extended example: $X_1, \ldots, X_n$ indep. $Ber(p)$, $Y_1, \ldots, Y_n$ indep. $Ber(p + \delta_i)$, with $\delta_i < c(p)$ small, then

$$d_{TV}(\mathcal{L}(X), \mathcal{L}(Y)) \leq C(p) \|\delta\|_2.$$

Proof is very similar.

Application to first-passage percolation

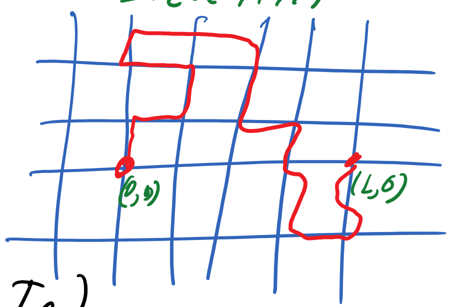
Consider first-passage perc. on $\mathbb{Z}^2$ with edge weights $\{a, b\}$, with $0 < a < b < \infty$ and $\mathbb{P}(\text{weight} = a) < p_c(\mathbb{Z}^2) = \frac{1}{2}$, edge weights ...

and $P(\text{weight}=a) < \underbrace{p_c(\mathbb{Z}^2)}_{\text{critical prob. for Bernoulli bond perc.}} = \frac{1}{2}$,

say $P(\text{weight}=a) = \frac{1}{3}$.

Let T_L be the passage time from $(0,0)$ to $(L,0)$.

Claim: $\operatorname{var}(T_L) \geq c \operatorname{Log}(L)$ (Newman-Piza 1994)



Idea of proof: Let (η_e) be the edge weights.

Define $(\tilde{\eta}_e)$ by $\tilde{\eta}_e = \min(\eta_e, I_e)$,

where (I_e) are indep., taking values in $\{a,b\}$,

with $P(I_e = a)$ chosen so that

$$P(\tilde{\eta}_e = a) = \frac{1}{3} + \delta_e \quad (\text{where } P(\eta_e = a) = \tfrac{1}{3}),$$

and with δ_e depending only on $|e|$ = dist. of e from $(0,0)$.

Let $\tilde{T}_L$ be the passage time from $(0,0)$ to $(L,0)$ with edge weights $(\tilde{\eta}_e)$.

Obvious that $\tilde{T}_L \leq T_L$ since $\tilde{\eta}_e \leq \eta_e$ for all e.

We will use that ($\delta_r = \delta_e$ for all e with $|e| = r$)

$$P\Big(T_L - \tilde{T}_L \geq \underbrace{c(a,b)\sum_{r=1}^{L}\delta_r}_{=:t}\Big) \geq c > 0$$

for some universal c.

This is a non-trivial fact, using that the a-weights don't percolate (since $P(\eta_e = a) < p_c(\mathbb{Z}^2)$) and in fact there must be a b-weight on the geodesic every constant number of steps (on average, since the size of ... of a-weights has exponential

steps (on average, since the size of connected comp. of a-weights has exponential tail).
(This may use the specific choice of δ below in which the δ's don't change quickly)

Consequently,

$$\mathbb{P}(|T_L - \mathbb{E}(T_L)| \ge \tfrac{t}{2}) + \mathbb{P}(|\tilde{T}_L - \mathbb{E}(T_L)| \ge \tfrac{t}{2}) \ge$$

$$\ge \mathbb{P}(T_L - \tilde{T}_L \ge t) \ge c > 0.$$

But $\mathbb{P}(|\tilde{T}_L - \mathbb{E}T_L| \ge t) \le \mathbb{P}(|T_L - \mathbb{E}T_L| \ge t) + d_{TV}(\mathcal{L}(\eta), \mathcal{L}(\tilde{\eta}))$

$\Rightarrow$ If $d_{TV}(\mathcal{L}(\eta), \mathcal{L}(\tilde{\eta})) \le \frac{c}{2}$

then $\mathbb{P}(|T_L - \mathbb{E}T_L| \ge \frac{t}{2}) \ge \frac{c}{4}$.

$\Rightarrow \operatorname{std}(T_L) \ge c' t.$

We reach the optimization problem:

$$\max t = \sum_{r=1}^{L} \delta_r$$

extended example

Subject to $d_{TV}(\mathcal{L}(\eta), \mathcal{L}(\tilde{\eta})) \le C \|\delta\|_2 \le C \left(\sum_{r=1}^{L} r \delta_r^2\right)^{\frac{1}{2}}$ being at most $\frac{c}{2}$.

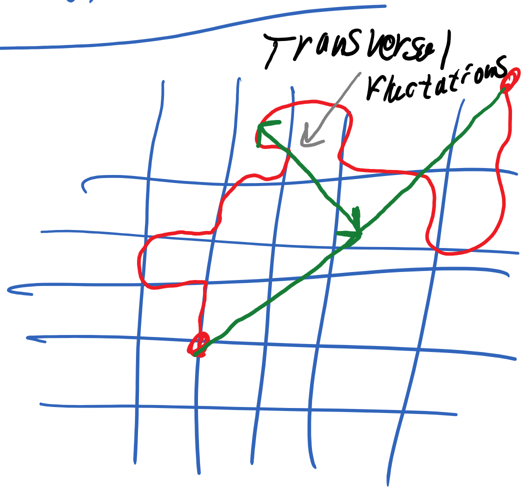
$\Rightarrow$ taking $\delta_r = \dfrac{c''}{r\sqrt{\log L}}$ gives $t = c''' \sqrt{\log L}$.

Open question: Does $\operatorname{Var}\begin{pmatrix}\text{passage time from origin} \\ \text{to } (L, 0, \dots, 0)\end{pmatrix} \xrightarrow[L\to\infty]{} \infty$ in dimension d (on $\mathbb{Z}^d$) for some $d \ge 3$?

Open question: Prove that the distance on the geodesic from the origin to $(L, 0, \dots, 0)$

Open question: Prove that the distance of the geodesic from the origin to $(L,0,\ldots,0)$ from a straight line is $o(L)$ in $\mathbb{Z}^d$? Find a quantitative $o(L)$ in this.

Transversal fluctuation exp., curvature of the limit shape and scaling relation



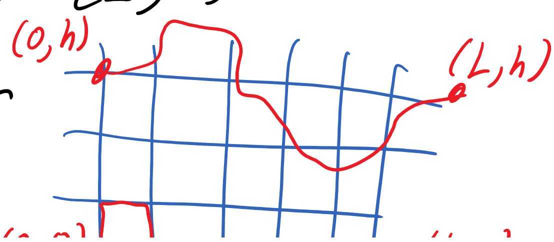
It is predicted that the deviation of the geodesic from a straight line path is of order L^{ξ}, for ξ called the **transversal fluctuation exponent**, for any endpoint of distance L from the origin.

In $d=2$, should have $\xi = \frac{2}{3}$.

Claim (Licea-Newman-Piza 1995, related to Wehr-Aizenman 1990): $\xi \geq \frac{1}{d+1}$ in $d \geq 2$.

In words, for any endpoint at dist. L, there is uniformly pos. prob. for some point on the geodesic to be at dist. $c L^{\frac{1}{d+1}}$ from the straight line path.

Idea of proof: Consider geodesic P_0 from origin to $(L,0,\ldots,0)$ and another geodesic P_h from $(0,h,0,\ldots,0)$ to $(L,h,0,\ldots,0)$.



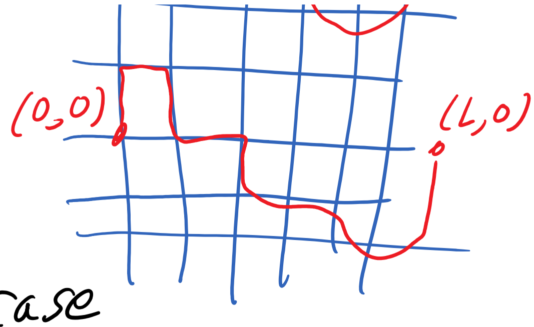
Let T_0 and T_h be their passage times.

Let T_0 … $_h$

passage times."

Claim: With high prob,

$$|T_0 - T_h| \le C \cdot h.$$



E.g., this is always the case for bounded weights. like we did for Bernoulli: weights in the prev. proof.

Consider "reducing the weights" in a tube of size $\frac{h}{2}$ around the straight line path from $(0,h,0,\ldots,0)$ to $(L,h,0,\ldots,0)$.

The number of edges in the tube is $\approx L \cdot h^{d-1}$. Can reduce each edge by $\frac{1}{\sqrt{L \cdot h^{d-1}}}$ ← so that ℓ^2-norm of reductions is ≈ 1. and still be with constant total variation dist.

If the path P_h has at least $L/2$ edges in the above tube, then after the reduction of the weights, the new passage time $\tilde{T}_h$ will satisfy $T_h - \tilde{T}_h \ge \frac{L}{2} \cdot \frac{1}{\sqrt{L \cdot h^{d-1}}} = \frac{1}{2}\sqrt{\frac{L}{h^{d-1}}}$.

But if the path P_0 does not intersect the tube, neither before nor after the reduction, then $T_0 = \tilde{T}_0$. However, we should have both

$$|T_0 - T_h| \le C h \quad \text{and} \quad |\tilde{T}_0 - \tilde{T}_h| \le C h.$$

Putting everything together, we get

$$\frac{1}{2}\sqrt{\frac{L}{h^{d-1}}} \le c_1 \cdot h \Rightarrow h \ge c L^{\frac{1}{d+1}}.$$

In conclusion, for any $h \ll L^{\frac{1}{d+1}}$, it must be that either $\mathcal{P}_h$ has less than $\frac{L}{2}$ edges in the tube or $\mathcal{P}_0$ intersects the tube before or after the weight reduction.

Since the weight reduction doesn't change prob. by much (d_{TV} close to 0) we see that a transversal fluc. of h occurs for either $\mathcal{P}_0$ or for $\mathcal{P}_h$.

Another exponent: It is predicted that the standard deviation of the passage time from the origin to a point at dist. L is of order L^{χ}.

The prev. argument shows also that

$$\chi \ge \frac{1}{2}(1-(d-1)\xi).$$

Scaling relation: It is predicted that

$$\chi = 2\xi - 1 \text{ in every dimension.}$$

In $d=2$, this together with $\chi \ge \frac{1}{2}(1-\xi)$ gives $\xi \ge \frac{3}{5}$. (Licea-Newman-Piza actually prove $\xi \ge \frac{3}{5}$ in $d=2$, but for a point to hyperplane version of the exponent).

for a point to hyperplane version of the exponent).

Why is the scaling relation expected:

It should follow from strict convexity of the limit shape.

Recall that the norm $\mathcal{N}$ of the first passage perc. is defined by

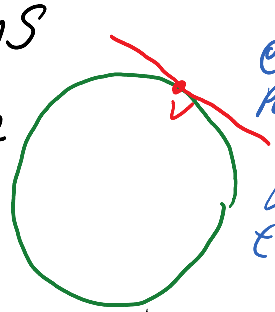
$$\mathcal{N}(z) = \lim_{n\to\infty}\left(\frac{1}{n}\ \text{time to go from origin to } n\cdot z\right)$$

The limit shape B is the unit ball of $\mathcal{N}$.

Strict convexity in direction v, where v is a vector in the boundary of the limit shape, means

$$c\|z\|_2^2 \le \mathcal{N}(v+z)-\mathcal{N}(v) \le C\|z\|_2^2$$

where $v+z$ is in the tangent hyperplane to B at v



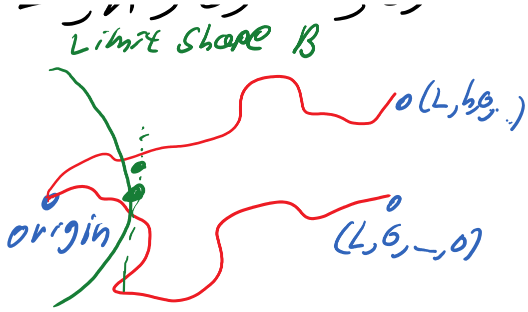
Open question: verify this for some weight dist., in all directions v.

Expected, e.g., for cont. weight dist. (with suitably finite moments)

Idea for scaling relation from strict convexity:

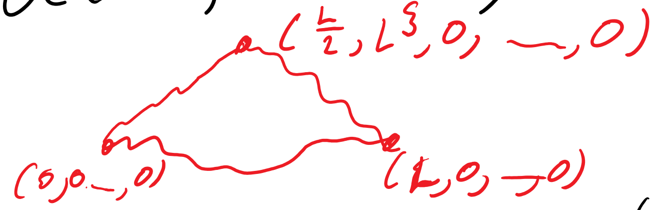
Consider geodesics from origin to $(L, 0, 0, \ldots)$ and to $(L, h, 0, \ldots, 0)$

$(L,0,0,\dots)$ and to $(\dots,h,0,\dots)$
with $h \ll L$.
What is the diff. in their passage times?



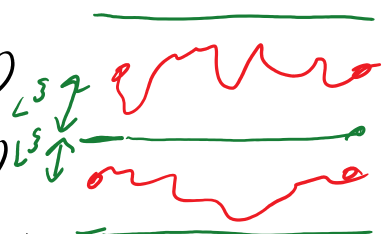
Assuming it behaves like the norm μ (ignoring non-random fluc. $\mathbb{E}(T_L)-\mu(L,0,\dots,0)$ and the random fluc. of order L^χ) we get $L\cdot\left(\frac{h}{L}\right)^2=\frac{h^2}{L}$.

Consequently, for a L^ξ deviation to occur, we may compare the paths



The path with the deviation will "cost" more than the straight line by $\frac{L^{2\xi}}{L}=L^{2\xi-1}$ (prev. calc. with $h=L^\xi$)
This should be compensated by the standard deviation of the passage time of order L^χ. So $\chi \geq 2\xi-1$.
For other direction, ← rough sketch suppose paths don't deviate from the straight line by more than L^ξ. Consider the geodesics

$P_0=(0,0,\dots,0)\longrightarrow(L,0,\dots,0)$
$P_{L^\xi}=(0,2L^\xi,\dots,0)\longrightarrow(L,2L^\xi,\dots,0)$



So all these geodesics use weights

$\Psi_{L^\xi} = (0, 2L^\xi, -, 0)$

Then these geodesics use weights in disjoint tubes of height L^ξ so their $\underbrace{\text{passage times}}_{T_0 \text{ and } T_{L^\xi}}$ are (almost) independent.

Thus $L^{2\chi} = Var(T_0) \approx \mathbb{E}(T_0 - T_{L^\xi})^2$

Then one shows that $T_0 - T_{L^\xi}$ is at most of order $L^{2\xi-1}$ by connecting the two geodesics at this cost.